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THE LINEAR DISTORTION OF FM SIGNALS IN BAND-PASS FILTERS FOR LARGE MODULATION FREQUENCIES

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The following article sets out to show that the linear distortion of a frequency-modulated (FM) harmonic signal in a symmetrical band-pass filter follows almost exactly the static response curve of the network provided that the modulation frequency is not less than about two-thirds of the semi-bandwidth.

Introduction

ONLY a very small portion of the field of distortion of FM signals in passive networks is covered in what follows. On the other hand, the results obtained below are to be considered as supplementary to those published in the following article.[1] In spite of that, it was decided to publish this article separately, as the mathematical techniques used here are quite distinct and show an obvious method of attack not previously used.

From the practical point of view, the problem would be stated as follows: What is the distortion (i.e., the ratio of the output to the input signal) to be expected when a FM signal of the form

$$i_{\text{in}} = i_0 \exp(j\omega_0 t + jm \sin pt) \tag{1}$$

is passed through a band-pass filter with a general admittance function

$$\begin{aligned} Y(\omega) &= G(\omega) + jB(\omega) \\ &= |Y(\omega)| \exp\{j\phi(\omega)\} \end{aligned} \tag{2}$$

where i_0 amplitude of the signal.

ω_0 $2\pi \times$ carrier frequency.

p $2\pi \times$ modulation frequency.

$m = \frac{\Delta}{p} =$ modulation index.

$\Delta = 2\pi \times$ maximum deviation frequency.

$Y =$ complex admittance.

$G =$ conductance.

$B =$ susceptance.

$|Y| =$ modulus of the admittance function.

$\phi =$ phase of the admittance function.

It is important to state here clearly what is meant by the expression "the output signal," when used by a practical engineer. The signal obtained at the output terminals of a passive network is both amplitude and frequency modulated. The amplitude modulation is subsequently equalised and the signal which is now of constant amplitude but varying phase is passed through a discriminator whose output is proportional to the instantaneous frequency of the signal. Thus, from the practical point of view, it is the derivative of the output phase with respect to time, which is of immediate interest.

Hence, there is not much point in presenting the output signal both as an amplitude and phase modulated time variable, unless the instantaneous frequency of that expression can be readily obtained.

In general, a solution of such a problem would thus present immense mathematical and, what is still more important, computational difficulties.

The distortion is most conveniently expressed in terms of the modulation radian frequency p and the maximum deviation radian frequency Δ. Further, it should be divided into two distinct parts:—

(1) Linear distortion, being that part of the output signal which is of the same frequency as the input modulation frequency.

(2) Non-linear distortion, being those components of the output signal which are various harmonics of the input modulation frequency.

It should be noted that here and subsequently, the expression "output signal" indicates, unless otherwise stated, the output from the linear frequency discriminator, as explained above.

In the present work, the range of the modulation radian frequency p and of the maximum deviation radian frequency Δ is limited; in order to fix the limits properly consider the ratios of p and Δ to the semi-bandwidth radian frequency of the network ω_B. We shall now divide the practical values of p arbitrarily into "large" and "small," using the following convention:—

p is said to be "large" if

$$0{\cdot}6 < \frac{p}{\omega_B} \leqslant 1 \tag{3}$$

and "small" if

$$0 < \frac{p}{\omega_B} < 0{\cdot}6 \tag{3a}$$

This mode of division will become clear below, where we shall see that the following analysis is applicable to "large" values of p only. On the other hand, the possible values of Δ cover the whole of the semi-bandwidth:

$$0 < \frac{\Delta}{\omega_B} < 1 \tag{4}$$

Furthermore, the analysis will only hold for highly selective networks, i.e., those where $Q = \frac{\omega_0}{2\omega_B}$ is sufficiently large to justify the use of arithmetical approximation. Thus, the frequency response of a general band-pass filter can be always expressed in terms of the parameter

$$X = \frac{\omega_0}{2\omega_B}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \tag{5}$$

When $\omega_0 \gg 2\omega_B$, i.e., $Q \gg 1$, we can write

$$X \doteqdot \frac{\omega - \omega_0}{\omega_B} \tag{5a}$$

so that X is proportional to the frequency deviation from the carrier point. Since the amplitude response of a band-pass filter is always an even function of X, the approximation (5a) will cause this response to be an even function of the frequency deviation about the carrier frequency.

It will be seen below that the use of (5a) will greatly facilitate the mathematical analysis.

Summing up, the following analysis is applicable only to highly selective band-pass filters with modulation frequency of the applied signal being not less than about two-thirds of the semi-bandwidth; the maximum deviation frequency can have any value within the semi-bandwidth.

The Development of the Method

Assuming the input signal to be a sinusoidally frequency modulated current as in (1), we can expand it into harmonics in the usual way, as follows:—

$$\begin{aligned} i_{in} &= i_0 \exp(j\omega_0 t + jm \sin pt) \\ &= i_0 \exp(j\omega_0 t) \sum_{n=-\infty}^{n=+\infty} J_n(m) \exp(jnpt) \end{aligned} \tag{6}$$

where $J_n(m)$ is the Bessel function of order n and argument m.

Let the transfer admittance of the band-pass filter be $Y(np)$, such that

$$\omega_B X = np = \omega - \omega_0 \tag{7}$$

Thus np is equal to the frequency deviation from the carrier, provided that we can use the arithmetical approximation (5a).

The voltage on the output terminals of the filter will be thus given by the Fourier series:—

$$e_{out} = i_0 \exp(j\omega_0 t) \sum_{n=-\infty}^{n=+\infty} J_n(m)\, Y(np) \exp(jnpt) \tag{8}$$

From (2)

$$Y(np) = G(np) + jB(np) \tag{2}$$

Hence, (8) becomes

$$e_{\text{out}} = i_0 \exp(j\omega_0 t) \sum_{n=-\infty}^{n=+\infty} J_n(m) \Big\{ \Big[G(np) \cos npt - B(np) \sin npt \Big] + j \Big[B(np) \cos npt + G(np) \sin npt \Big] \Big\} \tag{9}$$

Since the admittance function $Y(np)$ is assumed to be symmetrical round the carrier frequency:

$$\begin{aligned} G(-np) &= G(np) \\ B(-np) &= -B(np) \end{aligned} \tag{10}$$

Also

$$J_{-n}(m) = (-1)^n J_n(m) \tag{11}$$

The relations (10) and (11) greatly simplify further analysis; making use of them:—

$$\begin{aligned} e_{\text{out}} = i_0 \exp(j\omega_0 t) \Big\{ & J_0(m) \\ & + 2 \sum_{n=1}^{\infty} J_{2n}(m) \Big[G(2np) \cos 2npt - B(2np) \sin 2npt \Big] \\ & + 2j \sum_{n=1}^{\infty} J_{2n-1}(m) \Big[G(\overline{2n-1}p) \cos \overline{2n-1}pt + B(\overline{2n-1}p) \sin \overline{2n-1}pt \Big] \Big\} \end{aligned} \tag{12}$$

As stated in the introduction we shall require the phase of this expression; this can be written:—

$$\chi = \omega_0 t + \tan^{-1} \left\{ \frac{\sum_{n=1}^{\infty} \alpha_{2n-1}(m, p, t)}{1 + \sum_{n=1}^{\infty} \beta_{2n}(m, p, t)} \right\} \tag{13}$$

where:

$$\begin{aligned} \alpha_s(m, p, t) &= g_s(m, p) \sin spt + b_s(m, p) \cos spt \\ \beta_s(m, p, t) &= g_s(m, p) \cos spt - b_s(m, p) \sin spt \end{aligned} \tag{14}$$

and

$$\begin{aligned} g_s(m, p) &= \frac{2J_s(m)\, G(sp)}{J_0(m)} \\ b_s(m, p) &= \frac{2J_s(m)\, B(sp)}{J_0(m)} \end{aligned} \tag{15}$$

For brevity we shall write:—

$$\begin{aligned} \alpha_s(m, p, t) &\equiv \alpha_s \\ \beta_s(m, p, t) &\equiv \beta_s \end{aligned} \tag{16}$$

Note also that

$$\begin{aligned} \frac{\partial \alpha_s}{\partial t} &\equiv \alpha_s' = ps\beta_s \\ \frac{\partial \beta_s}{\partial t} &\equiv \beta_s' = -ps\alpha_s \end{aligned} \tag{17}$$

Expressions (15) show that provided m is sufficiently small (i.e., $m << 2{\cdot}405$, which is the first root of $J_0(m) = 0$), the values of $g_s(m, p)$ and $b_s(m, p)$ will decrease rapidly as s increases. We shall discuss below more fully the rate of decrease of these quantities where it will be seen that for a limited range of p and Δ it is sufficient to take into consideration only the first two terms.

Then

$$\chi \doteq \omega_0 t + \tan^{-1}\left\{\frac{\alpha_1 + \alpha_3}{1 + \beta_2}\right\} \tag{18}$$

Hence, the instantaneous frequency becomes:—

$$\omega_{\text{inst}} = \frac{\partial\chi}{\partial t} = \omega_0 + \frac{(1 + \beta_2)(\alpha_1' + \alpha_3') - \beta_2'(\alpha_1 + \alpha_3)}{(1 + \beta_2)^2 + (\alpha_1 + \alpha_3)^2}$$

$$= \omega_0 + p\,\frac{(1 + \beta_2)(\beta_1 + 3\beta_3) + 2\alpha_2(\alpha_1 + \alpha_3)}{(1 + \beta_2)^2 + (\alpha_1 + \alpha_3)^2} \tag{19}$$

from (17). This expression will be further simplified by neglecting higher order terms which are comparatively small. Then

$$\omega_{\text{inst}} - \omega_0 = p\,\frac{\beta_1 + 3\,\beta_3 + \beta_1\,\beta_2 + 2\,\alpha_1\,\alpha_2}{1 + 2\,\beta_2 + \alpha_1{}^2} \tag{20}$$

Substituting from (14) and making the necessary trigonometrical manipulations, we can group the terms in (18) in the following way:—

$$\omega_{\text{inst}} - \omega_0 = p\,\frac{N(pt)}{D(pt)} \tag{21}$$

where

$$\begin{aligned} N(pt) &= A_1 \cos pt + B_1 \sin pt + A_3 \cos 3pt + B_3 \sin 3pt \\ D(pt) &= A_0 + A_2 \cos 2pt + B_2 \sin 2pt \end{aligned} \tag{22}$$

and where

$$\left.\begin{aligned} A_0 &= 1 - \frac{1}{2}(g_1^2 + b_1^2) \\ A_1 &= g_1 - \frac{3}{2}(g_1\,g_2 + b_1\,b_2) \\ -B_1 &= b_1 - \frac{3}{2}(g_1\,b_2 + b_1\,g_2) \\ A_2 &= 2g_2 - \frac{1}{2}(g_1^2 - b_1^2) \\ -B_2 &= 2b_2 - g_1\,b_1 \\ A_3 &= 3g_3 - \frac{1}{2}(g_1\,g_2 - b_1\,b_2) \\ B_3 &= 3b_3 + \frac{1}{2}(g_1\,b_2 + b_1\,g_2) \end{aligned}\right\} \tag{23}$$

The quantities

$$\begin{aligned} g_s &\equiv g_s(m, p) \\ b_s &\equiv b_s(m, p) \end{aligned} \tag{24}$$

are defined in (15) as functions of the network response and the appropriate Bessel coefficients. The output signal is proportional to the time dependent part of the instantaneous frequency. Thus, the required value of the output current is:—

$$I = \frac{pN(pt)}{D(pt)} \tag{25}$$

This expression will consist of a signal of frequency $p/2\pi$ and a series of harmonics of that frequency. These components can all be separated by standard Fourier analysis. Thus, the nth harmonic of (25) becomes:—

$$I_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{pN(pt)}{D(pt)} \exp(jnpt)\, d\, pt \tag{26}$$

$$= \frac{p}{j\pi a_2^*} \oint_c \frac{(a_3^* Z^6 + a_1^* Z^4 + a_1 Z^2 + a_3)}{\left(Z^4 + \frac{2a_0}{a_2^*} Z^2 + \frac{a_2}{a_2^*}\right)} Z^{n-2}\, dZ \tag{27}$$

where $Z = \exp(jpt)$ and c denotes the unit circle. Also

$$\begin{aligned} a_s &= A_s + jB_s \\ a_s^* &= A_s - jB_s \end{aligned} \tag{28}$$

The roots of the denominator in (27) are given by:—

$$\delta^2 = -\frac{a_0}{a_2^*}(1 - \xi) \tag{29}$$

$$\varepsilon^2 = -\frac{a_0}{a_2^*}(1 + \xi) \tag{30}$$

where

$$\xi = \left(1 - \frac{a_2 a_2^*}{a_0^2}\right)^{\frac{1}{2}} \tag{31}$$

Hence (27) can be written

$$I_n = \frac{p}{j\pi a_2^*} \oint_c \frac{Q(Z) \,.\, Z^{n-2}}{(Z^2 - \varepsilon^2)(Z^2 - \delta^2)} dZ \tag{32}$$

where

$$Q(Z) = Q(-Z) = a_3^* Z^6 + a_1^* Z^4 + a_1 Z^2 + a_3 \tag{33}$$

The integrand in (32) has poles within the unit circle at $Z = 0$ (when $n < 2$) and $Z = \pm\,\delta$. Hence, the amplitude of the fundamental component of the output signal will be given by

$$I_1 = \frac{2p}{\delta^2 a_2^*}\left\{\frac{a_3}{\varepsilon^2} + \frac{Q(\delta^2)}{\delta^2 - \varepsilon^2}\right\}$$

$$= \frac{p}{\xi a_0}\left[a_1 + \delta^2(a_3^* \delta^2 + a_1^*) - \frac{a_3 a_2^*}{a_0(\xi + 1)}\right] \tag{34}$$

In a similar way we could deduce the expressions for further harmonics but this will not be done here since they will be well outside the pass-band of the network in the range of p and Δ where (21) is applicable; this is discussed below.

The Range of Applicability of the Results

In deriving expression (20) from (13) we have neglected in the total output phase expression (18) all terms multiplying harmonics higher than the third. Hence the range of the modulation index m will be determined by the relative magnitudes of the coefficients g_s and b_s defined in (15). An inspection of tables of Bessel functions will show us that for $m = 1$

$$\frac{J_1(m)}{J_4(m)} \doteq 180$$

so that we shall specify the range of m as

$$0 \leqslant m \leqslant 1 \qquad (35)$$

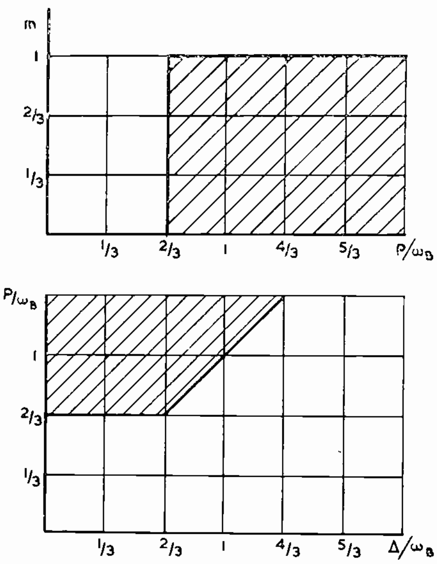

FIG. 1

The regions where equation (34) holds are shaded

Further, since we neglect all harmonics higher than third, the modulation frequency p must be sufficiently high, so that all its harmonics higher than the third should fall well beyond the pass-band of the network. This condition will obviously be governed by the selectivity of a particular band-pass filter, but we can safely propose a convenient range for p

$$\frac{2}{3} \leqslant \frac{p}{\omega_B} \qquad (36)$$

Thus, for a conventional maximally flat amplitude, triple tuned circuit, the attenuation associated with the fourth harmonic for $p = 2/3$ is about 25 db (for $Q = 20$), while for a single tuned circuit the corresponding attenuation is only 9 db. Hence, the condition (36) will hold well for comparatively selective networks. In Fig. 1 we see the shaded regions of p/ω_B, Δ/ω_B and m where inequalities (35) and (36) hold.

The Fundamental Component of the Output Signal

Expression (34) can be directly computed for any type of network whose admittance is symmetrical round the carrier point. It will be useful, however, to obtain the first order approximation of (34) in terms of the network and signal parameters.

Remembering that for " large " values of p

$$|a_3| < |a_2| < |a_1| < 1 < |a_0|$$

we obtain

$$I_1 \doteq p \left[\frac{a_1}{a_0} - \frac{1}{2}\frac{a_2\, a_1^*}{a_0^2}\right] \qquad (37)$$

$$\doteq p\, a_1 \left[1 - \frac{1}{2}|a_1|^2 - \frac{1}{2}\, a_2 \exp\left(-2j\gamma_1\right)\right] \qquad (38)$$

where η_1 is the phase of a_1 in (28). Hence, from (28), (23) and (15)

$$I \propto Y^*(p) \left\{ 1 + \frac{1}{8} m^2 \left[1 - 4\,|Y(p)|^2 - 2\,|Y(2p)| \exp\left(j\left[2\phi(p) - \phi(2p)\right]\right) \right] \right\} \tag{39}$$

We observe that for small values of Δ and sufficiently high p, the fundamental component follows the static characteristic $Y(p)$. In particular, the amplitude of this component is governed by the static amplitude response, while its phase is the conjugate of the static phase.

We can thus state that the attenuation of the fundamental is given by:—

$$D(p, \Delta) \doteqdot 20\,[\log_{10} Y(p) + \log_{10} A(p, \Delta)] \tag{40}$$

while the phase is

$$\theta(p, \Delta) \doteqdot 2\pi - \phi(p) + \psi(p, \Delta) \tag{41}$$

where

$$A(p, \Delta) \doteqdot 1 + \frac{1}{8} m^2 \left\{ 1 - 4\,|Y(p)|^2 - 2\,|Y(2p)| \cos\left[2\phi(p) - \phi(2p)\right] \right\} \tag{42}$$

$$\tan[\psi(p, \Delta)] \doteqdot -\frac{\frac{1}{4} m^2\,|Y(2p)| \sin\left[2\phi(p) - \phi(2p)\right]}{A(p, \Delta)} \tag{43}$$

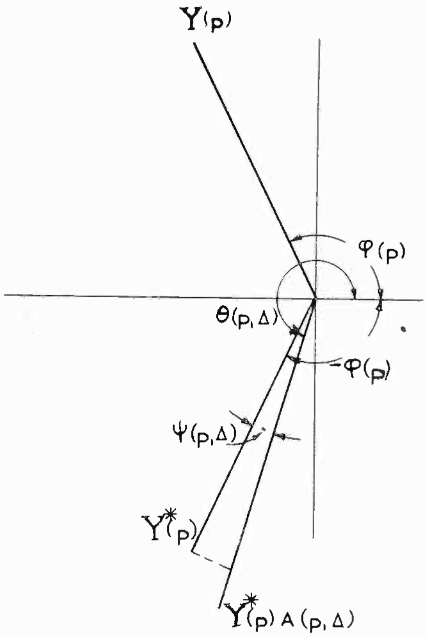

FIG. 2

Vector relation between the output signal and the static response (not to scale)

Fig. 2 shows the relations between the static response and the fundamental component of the output, as expressed in terms of the correction factors $A(p, \Delta)$ and $\psi(p, \Delta)$. From (42) it is seen that for $p/\omega_B < 1$, the amplitude correction factor $A(p, \Delta) < 0$ since then $|Y(p)|^2 > \frac{1}{4}$. Thus, the attenuation curve of the output signal will, for modulation frequencies less than the semi-bandwidth, lie under the static response curve; for modulation frequencies greater than the semi-bandwidth, the attenuation of the output signal may, however, be smaller than the corresponding static attenuation. In general, however, both correction factors (42) and (43) are quite small. We shall now apply these results to two typical networks, namely, the single tuned one and the maximally flat amplitude triple tuned one (Butterworth type).

Applications

For sufficiently high Q, the transfer admittance of a single tuned circuit may be written:—

$$Y(nP) = (1 + n^2P^2)^{-\frac{1}{2}} \exp\{-j \tan^{-1}(nP)\} \tag{44}$$

$$= \frac{1}{1 + n^2P^2} - j\,\frac{nP}{1 + n^2P^2} \tag{44a}$$

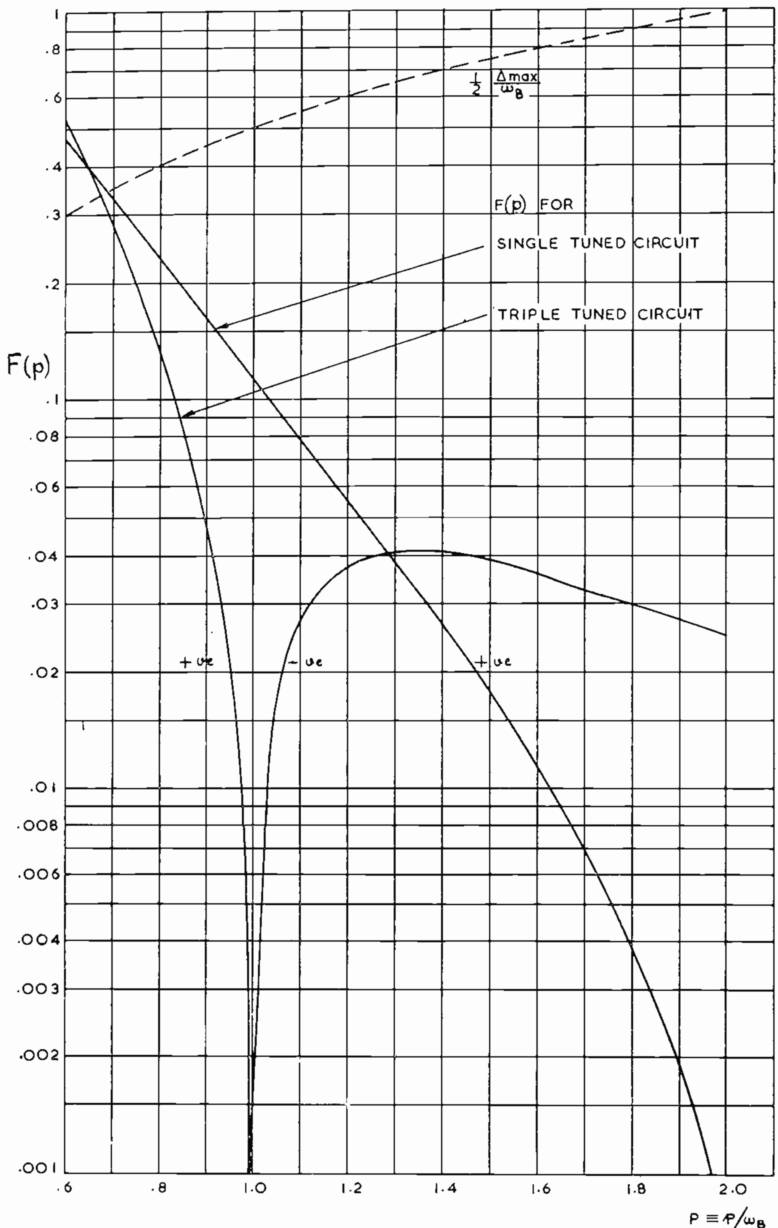

FIG. 3
The distortion correction factor for two networks discussed in the text

while for a conventional triple tuned, maximally flat amplitude circuit (see Ref. 1):—

$$Y(nP) = (1 + n^6 P^6)^{-\frac{1}{2}} \exp\left\{-j \tan^{-1}\left(\frac{2nP - n^3 P^3}{1 - 2n^2 P^2}\right)\right\} \tag{45}$$

$$= \frac{1 - 2n^2 P^2}{1 + n^6 P^6} - j\,\frac{2nP - n^3 P^3}{1 + n^6 P^6} \tag{45a}$$

The variable nP is to be considered here as dimensionless, since P is regarded as a fraction of the semi-bandwidth (see pp. 102-103):—

$$P = p/\omega_B.$$

Since $m = \Delta/P$, the amplitude correction factor $A(P, \Delta)$, as defined in (42) and in Fig. 2 can be written:—

$$A(P, \Delta) = 1 - \Delta^2 F(P) \quad \left(P \geqslant \frac{2}{3}\right) \tag{46}$$

where Δ here again is to be regarded as normalised with regard to the semi-bandwidth.

For the single tuned circuit (44)

$$F(P) = \frac{5 + 17P^2 - 4P^4}{8P^2(1 + P^2)(1 + 4P^2)} \tag{47}$$

while for the maximally flat amplitude, triple tuned circuit

$$F(P) = \frac{5 + 253P^6 - 48P^8 - 64P^{12}}{8P^2(1 + P^6)(1 + 64P^6)} \tag{48}$$

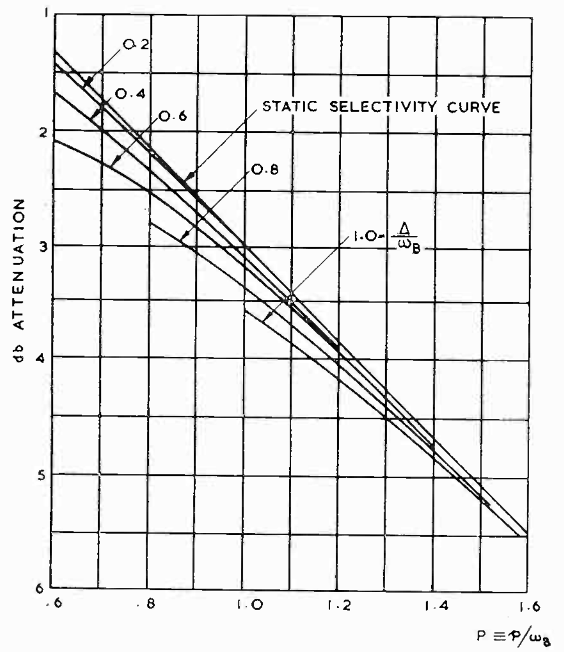

FIG. 4

Output signal attenuation for the single tuned circuit

These two functions (47) and (48) are shown in Fig. 3 for suitable range of the modulation frequency P normalised with respect to the semi-bandwidth (see Fig. 1). The dashed curve shows the maximum value of the normalised maximum deviation frequency for which the graphs are applicable for a given value of the modulation frequency P.

In Fig. 3 we see that for the modulation frequency $p \doteq 0{\cdot}98\omega_B$, the factor $F(P)$ for a triple tuned network changes sign and the attenuation of the fundamental component of the output signal becomes smaller than the static attenuation.

In Fig. 4 the static attenuation curve is compared with the attenuations of FM signals for various values of P and Δ in a single tuned circuit. We observe that this attenuation hardly ever exceeds the static attenuation by 1 db within the range of the modulation frequency where our formulae apply.

The corresponding curves for a maximally flat amplitude, triple tuned circuit

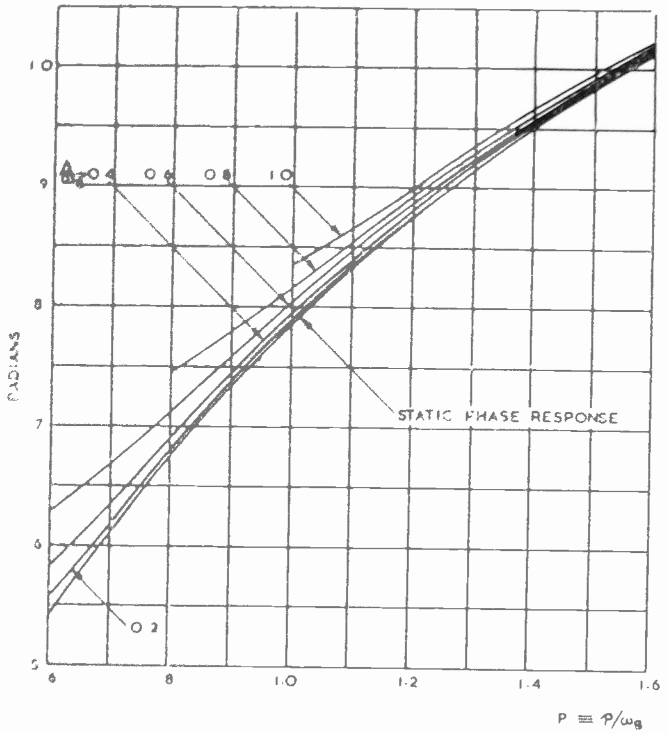

Fig. 5
Phase response for the single tuned circuit

would almost coincide with the static attenuation curve, as can be seen from the form of the corresponding $F(P)$ factor in Fig. 3.

This network will be discussed fully in the following article[1] where the corresponding distortion curves will be shown over the whole range of P in the pass-band.

It is sometimes important (e.g., in problems connected with the double FM modulation) to determine the phase of the output signal. This can be evaluated directly from equation (43) and is shown for the single tuned circuit in Fig. 5. We observe again here that the phase almost follows the static response curve. A similar set of curves for the triple tuned case will be shown in the following article[1].

Conclusion and Summary

Equations (40) and (41) show that the linear component of the output follows essentially the static network attenuation and phase response curves provided that

(1) The modulation frequency P is at least equal to two thirds of the semi-bandwidth.

(2) The modulation index m does not exceed unity.

(3) The network is sufficiently selective to warrant the use of the " arithmetic " approximation in its response equations:—

$$X = \frac{\omega_0}{2\,\omega_B}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \approx \frac{\omega - \omega_0}{\omega_B}$$

These conclusions were applied to a single tuned network with results shown in Figs. 4 and 5. They agree well with the experimental work performed in the laboratories of the Marconi Co. and also in U.S.A.[2].

It is proposed to continue this work with two aims in view:—

(1) To investigate the linear and non-linear distortion in band-pass filters for " small " values of the modulation frequency p.

(2) To generalise the results for networks where the " arithmetic " approximation no longer holds.

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THE DISTORTION OF FM SIGNALS IN PASSIVE NETWORKS

By R. H. P. Collings and J. K. Skwirzynski, B.Sc., A.R.C.S.

The distortion of sinusoidal FM signals in passive networks is examined and a new method of treating this distortion is derived which compares favourably with existing methods. The results are applicable within a well defined region of network and signal parameters. The method is applied to the maximally flat amplitude triple tuned circuit.

The results are exemplified by Tables and Graphs. Special attention is given to the effects of detuning.

Introduction

THE aim of this article is to present general results on the behaviour of sinusoidally frequency modulated signals in electric networks. It is hoped that this has been achieved completely within a certain range of the signal parameters; these are the modulation frequency and the maximum deviation frequency in relation to the network parameters such as the mid-band frequency and the bandwidth.

The results obtained are general in the sense that they can be applied to any passive network.

Part I discusses the existing methods of attack, especially that due to Fry and Carson[1]; a careful examination of these methods is given with special reference to their drawbacks.

It will be found that the so-called " quasi-stationary theory " is in many ways unreliable, especially from the point of view of the fundamental component of the output signal.

Part II gives the derivation of theoretical formulae together with a thorough discussion on the range of their applicability. The fundamental component and the first four harmonics of the output instantaneous frequency are then given in terms of a series expansion in the modulation frequency and the modulation index. The coefficients of expansion depend entirely on the network parameters. Since the latter were obtained by expanding the network functions in terms of the frequency deviation from the carrier, it was found necessary to investigate the possibility of such a simple expansion for band pass filters which are not symmetrical about the mid-band point. This forms the subject of Part III where a general table is given converting the coefficients of expansion of a network function in terms of the " band pass parameter ":

$$X = Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

to an expansion in terms of the frequency deviation $\delta\omega = \omega - \omega_0$. This table also includes the effects of a possible detuning.

Part IV contains a discussion of the maximally flat amplitude triple tuned circuit (also called the " Butterworth " circuit). The distortion coefficients of this network are given in a special table and the fundamental and harmonic distortions of the output instantaneous frequency are plotted for a selected range of parameters. The detuning of such network shows here interesting consequences. Furthermore, the distortion of the fundamental component is, for a "high Q" case, extended to modulation frequencies up to the value of one-half bandwidth by means of results obtained in the preceding article.

The final and fifth part of this article contains a comparison of our method with those discussed in Part I; it is found that it agrees well with the results obtained previously by Fry and Carson[1] and by Stumpers[2].

PART I

GENERAL CHARACTERISTICS OF EXISTING METHODS

The problem of estimating the distortion of *FM* signals by linear networks has received the attention of many eminent mathematicians and engineers; it was first considered by Fry and Carson[1] in their classic article and since it has been followed, up to now, by most other workers in this field, it will be useful to discuss their method in some detail.

1.1. The Fry and Carson Method

In their paper[1] Fry and Carson deduce the following expression (29) for the output current I due to the applied e.m.f. as defined in their equation (25):

$$I = E \exp \left(j\,\omega_c t + j \int_0^t \mu\, dt\right) \left\{ Y(j\,\omega_c) + \sum_{n=1}^{\infty} \frac{j^n}{n!} C_n\; Y^{(n)}(j\,\omega_c) \right\} \tag{29}$$

where $Y(j\,\omega)$ is the transfer admittance of the network, ω_c is the carrier angular frequency and $\mu = \mu(t)$ is the variable part of the instantaneous angular frequency.

Expression (29) forms the basis of most of the work done on this subject by Fry and Carson as well as by other authors. As Fry and Carson remark, however, the series in (29) may not be convergent and is very laborious to compute.

Actually the series in (29) is of asymptotic nature (as has been shown by Stumpers; see Ref. 2 and Section 1.2.1 below) and diverges in most practical cases. This is due to the peculiar structure of the operators C_n. Thus:

$C_1 = \mu$	$C_3 = \mu^3 - 3j\mu\mu'$	$- \mu''$
$C_2 = \mu^2 - j\mu'$	$C_4 = \mu^4 - 6j\mu^2\mu'$	$- 4\mu\mu'' - 3\mu'^2 + j\mu'''$

Fry and Carson then use the following approximation:—

$$C_n \doteq \mu^n - j\,\frac{(n-1)n}{2}\,\mu'\,\mu^{n-2} \quad \text{(p. 519)}$$

which as we see includes only the first two terms in each C_n; i.e., in the above table the terms which are to the left of the vertical dashed line.

Then (29) becomes:—

$$I = E \exp \left(j\,\omega_c\, t + j \int_0^t \mu\, dt\right)\, Y\,(j\,\omega_1\, t)$$

where

$$Y\,(j\,\omega_1\, t) = Y\,(j\Omega) + \frac{j\mu'}{2}\, Y^2\,(j\Omega) \tag{16a}$$

and where

$$\Omega(t) = \omega_c + \mu(t) \tag{3}$$

The expression (16a) is of great importance in the further history of the problem. Thus it is identical with the formulae used subsequently by Stumpers (Ref. 2), van der Pol (Ref. 3) and hence by all authors following van der Pol.

Many writers have been content to take the first term only of (16a); this is the so-called "quasi-stationary approximation." Some very useful results have been obtained by means of this approximation but there has always been some doubt as to the range of its validity.

From inspection of the table of coefficients C_n, we observe that the "quasi-stationary" term in (16a) includes only the first term of the expression for C_n (i.e., μ^n) while the second term in (16a) corresponds to the second term in the expression for C_n (i.e. $-j\frac{(n-1)n}{2}\mu'\mu^{n-2}$). Both first and second terms include distortion components and examination shows that the "quasi-stationary" term includes some distortion components cancelled by the components included in the second term. Furthermore, some of the remaining terms are again cancelled by those parts of the expression for C_n which are neglected in (16a). The use, either of the "quasi-stationary" approximation, or of (16a) thus results in the appearance of distortion components which would not be shown if the complete expression (29) were used.

It is thus obvious that the "quasi-stationary" approximation cannot give the correct output/input relation; in fact, it is only true for zero modulation frequency, a rather trivial case. For, let

$$\mu = \Delta \sin pt$$

where Δ - maximum deviation frequency,
p — modulation frequency.

Then the terms in the first column of the table of C_n become proportional to

$$\Delta, \Delta^2, \Delta^3, \ldots \Delta^n$$

and are completely independent of p. This fact, of course, does not preclude the output harmonics of the instantaneous frequency being functions of p, for if we expand $Y(j\Omega)$ in a power series of the frequency deviation from the carrier, p will appear there; such dependence, however, on the modulation frequency will not necessarily give the actual magnitude of the harmonics and the fundamental, especially of the latter.

Similarly, by including the second term in the expression (16a) we shall still obtain an answer true only as far as the first order in p; the terms in the second column in the table of C_n become proportional to

$$\Delta p, \Delta^2 p, \Delta^3 p, \ldots \Delta^n p.$$

One can thus state generally that the expression (16a) is only true as far as the first power of p (see section 1.1.2).

1.1.1. The Fry and Carson "Retarded Time" Formula

Fry and Carson include also in their paper a slightly modified approach, which, however, is of great importance, as will be seen below. They write

$$Y(j\omega_c + j\omega) = |Y(j\omega_c + j\omega)| e^{-j\varphi} \tag{33}$$

where

$$\varphi = \omega_c \tau + \omega \tau + \beta(\omega)$$
$$\beta(0) = \beta'(0) = 0$$

and where τ is the initial delay of the network.

$$I = E \exp \left(j \int_0^{t'} \Omega(\tau)\, d\tau \right)$$
$$\times \left[1 + j\mu' \left| Y(j\omega_c) \right| \Big|_0^{\prime} + \sum_{n=2}^{\infty} \frac{r_n}{n!} C_n(t') \right] \qquad (35)$$

where $t' = t - \tau$ is the " retarded " time, and

$$r_n = r_n(\omega_c) = \left\{ \frac{\partial^n}{\partial \omega_c^n} \left| Y(j\omega_c + j\omega) \right| e^{-j\beta(\omega)} \right\}_{\omega=0}$$

This expression has a great advantage in that it presents the output current in an easily interpreted way. An *FM* signal when passed through a network will be delayed by the amount τ and also distorted; the distortion as seen from (35) is due to the varying amplitude response of the network as well as to the non-linearity of phase as expressed by β.

It will be seen in the last part of this article (p. 134) that the " retarded time " formula of Fry and Carson agrees completely with that derived by us in an entirely independent way.

It is necessary to stress the fact that the " retarded time " formula of Fry and Carson is the only one met up to now in the literature which allows the engineer to obtain the required results without an excessive amount of mathematical labour.

We observe, however, that in (35) Fry and Carson do not include the " quasi-stationary " admittance at all. It will be interesting, therefore, to consider how the " quasi-stationary " admittance of a linear phase network reacts to a harmonic *FM* signal.

1.1.2. Critical Test of the " Quasi-stationary " Approximation

Suppose that

$$Y(j\omega) = e^{j\alpha\omega}$$

i.e., the network has a flat amplitude response and a linear phase response. Then, from 1.1. (16a)

$$Y(j\Omega) = e^{j\alpha\omega_c + j\alpha\mu}$$

Hence:

$$I = E \exp \left\{ j\omega_c t + j \int_0^t \mu\, dt + j\alpha\omega_c + j\alpha\mu \right\}$$

and the instantaneous frequency becomes:

$$\omega_i = \omega_c + \mu + \alpha\mu'$$

or, when

$$\mu = \Delta \sin pt$$
$$\omega_i = \omega_c + \Delta \sin pt + \alpha \Delta p \cos pt$$

Thus, the amplitude of the fundamental becomes proportional to:

$$\Delta \sqrt{1 + \alpha^2 p^2}$$

which is in complete disagreement with equation (35) of the Fry and Carson paper, unless $p = 0$. The reason for this discrepancy becomes obvious if we remember that $Y(j\Omega)$ was obtained by Fry and Carson by including only the first terms of the expression for C_n (see p. 114) which are not sensitive in p.

This example shows directly how dangerous it is to use the " quasi-stationary " approximation even for qualitative results for it would seem that the amplitude of the fundamental would increase with the modulation frequency which, as will be shown below, is not true; the amplitude of the fundamental would in this case remain constant, independent of p. This point is further discussed in section 5.2.

1.2. Prof. van der Pol's Method.

In his paper on " The Fundamental Principles of Frequency Modulation "(3) Prof. van der Pol derives the " quasi-stationary " approximation in a somewhat different way from Fry and Carson. The output current is given formally in an operational form:

$$i(t) = \exp\left\{j\omega t + js\left(t + \frac{d}{dj\omega}\right)\right\} Y(j\omega) \tag{36}$$

where

$$s(t) = \Delta\omega \int^{t} g(t)\, dt$$

and $g(t)$ is the modulating signal. Expression (36) is deduced from the expansion of a time dependent admittance which results in an asymptotic series. (See section 1.2.1. below.)

Considering only the first term of the expansion (36), Prof. van der Pol obtains the quasi-stationary solution:

$$i(t) = \exp\left(j\int^{t} \Omega\, dt\right)\, Y(j\Omega) \tag{39}$$

which he defines as such when " the circuit is completely capable of following through stationary states the variable frequency of the applied emf." The solution (39) thus represents a limiting case; but we have seen above (section 1.1.2.) that unless p vanishes identically, an approximation like (39) may lead to erroneous results. Thus Prof. van der Pol states that " in the quasi-stationary approximation the total current signal (including its distortion) is completely determined by the phase characteristic of the admittance only, and is therefore—at least explicitly—independent of the amplitude characteristic of the admittance." Applying this conclusion to the case treated above (section 1.1.2.) we again find that the output instantaneous frequency deviation is increased by:

$$1 + \tfrac{1}{2}\,\alpha^2\,(\Delta\omega)^2\,p^2$$

where α is the initial group delay of the admittance. Moreover, Prof. van der Pol states that the " quasi-stationary " approximation is true provided:

$$p\,\Delta\omega << \tfrac{1}{2}\,B^2 \qquad \text{(p.158)}$$

where: B is equal to the 3 db bandwidth of the network.

Such a condition is rather misleading for it imposes no restriction on the value of p alone but on the product $p\Delta\omega$. We shall see below that Stumpers modifies van der Pol's condition correctly (section 1.2.1.). It may be added, also, that in the

Fry and Carson expansion discussed above, the " quasi-stationary " approximation is intrinsically non-sensitive as far as p is concerned and van der Pol's limiting condition shows it perfectly.

Van der Pol discusses further the next term in his expansion, but since this is included in the Stumpers' formulation, it is better to describe it there.

1.2.1. The Stumpers' Method.

The paper of Dr. F. L. H. M. Stumpers on " Distortion of Frequency Modulated Signals in Electrical Networks " is the first critical contribution dealing with the problem. Stumpers recognises the asymptotic nature of the Fry and Carson series and concludes that "it is asymptotic in the sense of Poincaré when $p \to 0$." Hence the expression for the output current is expanded in terms of p and, what is still more important, the range of the validity of the expansion is defined by two conditions, one involving the product $p\Delta\omega$ (as in van der Pol's paper), the other involving p only (see p.86 of ref. 2).

The Stumpers' expression is very simple to use especially when the power expansion of the network function is given.

We shall discuss this method again in the last part of this article (p. 135) where the Stumpers' formula is compared with the Fry and Carson and our present formulation.

PART II

GENERAL DESCRIPTION OF THE METHOD

2.1. Theory

The input current is assumed to be:

$$I_{in} = I_0 \exp\{j\omega_0 t + jm \sin p\omega_B t\} \tag{2.1.1}$$

which can be expanded as a Fourier series:

$$I_{in} = I_0 \exp(j\omega_0 t) \sum_{n=-\infty}^{\infty} J_n(m) \exp(jnp\omega_B t) \tag{2.1.2}$$

Let also:

$$Z(\omega) = M(\omega) \exp\{j\varphi(\omega)\} \tag{2.1.3}$$

be the transfer impedance of a passive network. For the purpose of our analysis we shall express the independent variable ω in (2.1.3.) in terms of the modulation frequency $p\omega_B$ of our input signal (2.1.1.) by writing:

$$\omega = \omega_0 + np\omega_B \tag{2.1.4}$$

The output signal due to (2.1.1) expressed as the voltage on the output terminals of the network whose impedance is (2.1.3) will be given by:

$$E = I_0 \exp(j\omega_0 t) \sum_{n=-\infty}^{\infty} J_n(m)\, M(np\omega_B) \exp\{jnp\omega_B t + j\varphi(np\omega_B)\} \tag{2.1.5}$$

The amplitude M and the phase φ of (2.1.3) can be expanded round the carrier frequency as a series in the deviation from that frequency; the method of expansion is discussed at length in Part III of this article. For the time being let us write:

$$M(np\,\omega_B) = \sum_{s=0}^{\infty} b_s\,(np\,\omega_B)^s \tag{2.1.6}$$

$$\varphi(np\,\omega_B) = \sum_{s=0}^{\infty} a_s\,(np\,\omega_B)^s \tag{2.1.7}$$

In most practical cases the coefficient a_0 in (2.1.7), which represents the constant phase at the carrier frequency, will vanish; the coefficient a_1 represents here the initial group delay. It was already pointed out by Fry and Carson[1] (also see in Part I of this article, section 1.1.1), that it is convenient to exclude this delay from the analysis at this stage. This can be done by introducing the " delayed time " of the output signal:

$$\tau = t + a_1 \tag{2.1.8}$$

Then, substituting (2.1.6 — 8) in (2.1.5), we obtain

$$E = I_0 \exp\{j\,\omega_0\,(\tau - a_1) + ja_0\}\ \Xi \tag{2.1.9}$$

where

$$\Xi = \sum_{n=-\infty}^{\infty} J_n(m)\ M(np\,\omega_B)\ \exp\{jnp\,\omega_B\,\tau + j\Psi(np\,\omega_B)\} \tag{2.1.10}$$

and where:

$$\Psi\,(np\,\omega_B) = \varphi\,(np\,\omega_B) - a_0 - a_1\,np\,\omega_B \tag{2.1.11}$$

from (2.1.7).

It has been pointed out already in section 1.1.2 that an ideal network with flat amplitude response and linear phase response should introduce no distortion beyond shifting the time scale of the output signal by the amount equal to its delay. Thus in (2.1.10), if we put:

$$M\,(np\,\omega_B) \equiv 1$$

$$\Psi\,(np\,\omega_B) \equiv 0$$

we obtain:

$$\Xi = \exp\{jm \sin p\,\omega_B\,\tau\}$$

and:

$$E = I_0 \exp\{j\,\omega_0\,(\tau - a_1) + ja_0 + jm \sin p\,\omega_B\,\tau\}$$

as expected.

Returning to (2.1.10), consider the expansion of the product:

$$M\,(np\,\omega_B)\,\exp\{j\Psi\,(np\,\omega_B)\} \doteqdot \sum_{s=0}^{q} d_s\,(np\,\omega_B)^s \tag{2.1.12}$$

where q is a large but finite integer. The choice of q will obviously be governed by the number of terms sufficient to represent the impedance function within its passband and also—still more important—by the algebraical and computational labour involved in obtaining required results. In our analysis we have chosen $q = 6$; it will be shown in section 2.1.1 that such an expansion of the impedance function covers well the required ranges of frequency deviation, without making the amount of mathematical work prohibitive.

The coefficients d_s are easily found as polynomials in terms of a_s and b_s, the phase and amplitude coefficients of the impedance function respectively.

Hence substituting (2.1.12) in (2.1.10):

$$\Xi = \sum_{n=-\infty}^{\infty} J_n(m) \left\{ \sum_{s=0}^{q} d_s (np\omega_B)^s \right\} \exp (jnp\omega_B \tau) \tag{2.1.13}$$

Before proceeding with the analysis further it will be necessary to show under what conditions the series Ξ is an absolutely convergent one. The modulus of the n^{th} term of (2.1.13) becomes:

$$\xi_n = J_n(m) \left[d_0 + d_1 (np\omega_B) + \ldots d_q (np\omega_B)^q \right]$$

As n increases, while m is reasonably small, we can write:

$$J_n(m) \doteqdot (\tfrac{1}{2}m)^n \frac{1}{n!}$$

so that:

$$\frac{\xi_n}{\xi_{n-1}} = \frac{m}{2n} \left\{ \frac{d_0 + d_1 (np\omega_B) + \ldots d_q (np\omega_B)^q}{d_0 + d_1 (\overline{n-1}\, p\omega_B) + \ldots d_q (\overline{n-1}\, p\omega_B)^q} \right\}$$

Hence:

$$\lim_{n \to \infty} \left(\frac{\xi_n}{\xi_{n-1}} \right) = \frac{m}{2n} \left(1 + \frac{q}{n} \right)$$

It follows that provided q is finite, the above limit vanishes and thus the series (2.1.13) is absolutely convergent for any finite q and m. Note, however, that for a given value of q, the maximum values of p and m must be appropriately limited as explained in section 2.1.1.

Under these circumstances it is permissible to exchange the order of summation in (2.1.13):

$$\Xi = \sum_{s=0}^{q} d_s \left\{ \sum_{n=-\infty}^{\infty} J_n(m) (np\omega_B)^s \exp (jnp\omega_B \tau) \right\} \tag{2.1.14}$$

$$= \sum_{s=0}^{q} (-j)^s d_s \left(\frac{\partial}{\partial \tau} \right)^s [\exp jm \sin p\omega_B \tau)] \tag{2.1.15}$$

Now let:

$$\exp (jm \sin p\omega_B \tau) = \exp [jg(\tau)] = y(\tau) \tag{2.1.16}$$

where:

$$g(\tau) = m \sin p\omega_B \tau \tag{2.1.17}$$

Thus, substituting (2.1.15, 16) into (2.1.9), we obtain the final formula for the output voltage:

$$E = I_0 \exp \{ j\omega_0 (\tau - a_1) + ja_0 + jm \sin p\omega_B \tau \} \chi \tag{2.1.18}$$

where:

$$\chi = \sum_{s=0}^{q} (-j)^s d_s \left\{ \frac{1}{y(\tau)} \left(\frac{\partial}{\partial \tau} \right)^s [y(\tau)] \right\} \tag{2.1.19}$$

In the expression (2.1.18) the first part of the product represents an undistorted output signal subject only to a constant delay, as on p. 15. The second part, i.e., the sum χ, differs from unity to the extent to which distortion has been introduced by the network.

Subject to the limitations introduced by the convergence proof above (i.e., finite q), the output signal is completely specified by (2.1.18, 19). In practical applications the accuracy of the solution will be found to depend only on the extent to which the transfer impedance can be represented in the form (2.1.6, 7) by a finite number of terms.

2.1.1. Range of Applicability of the Formulae

We should note at this stage that for given value of q as defined in (2.1.19) there are further restrictions which must be imposed on the values of p and m owing to the nature of the expansions described in section 2.1. and leading to (2.1.19).

As mentioned on p. 119 the choice of q will be governed by the number of terms sufficient to represent the impedance function of a given network within as large a portion of the pass-band as possible without the necessary mathematical labour being prohibitive.

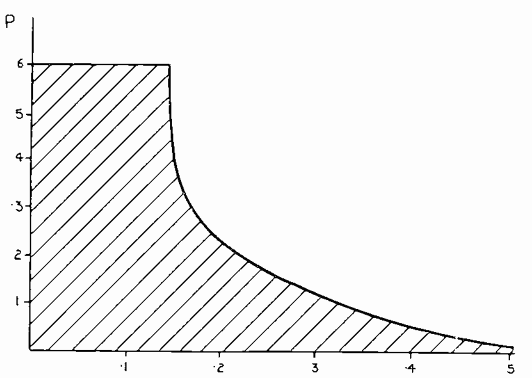


FIG. 1

Regions within which it is permitted to use the expansion (2.1.19) for $q = 6$

It was decided by the authors to take the value of $q = 6$ which in particular covers about 60% of the centre of the 3db band of a conventional maximally flat (Butterworth) triply tuned network. For a given value of the modulation index $m = \Delta/p$ we can easily find the number of sidebands not exceeding a certain small value, say 0·01 times the unmodulated amplitude of the carrier. Assuming that further sidebands are not significant we can find the maximum value of the normalised modulation frequency p for a given m (and hence Δ) such that no significant sidebands will lie outside the limits $\pm$ 0·6 of the semibandwidth where our expression ceases to be valid. Fig. 1 shows the resulting region of validity of p and Δ (shaded).

2.2. The General Expressions for the Instantaneous Frequency.

The direct application of the expression (2.1.19) to practical problems involves considerable labour; it is thus fortunate that the work can be reduced to quite modest proportions by some further simple though tedious manipulations.

These consist of evaluating the network coefficients d, of finding the time function appearing within the curly brackets of (2.1.19) in terms of harmonics of the delayed input signal and of collecting the terms contributing to each individual harmonic. In this way we obtain an expression in the form:

$$\chi = \exp\left(\mathcal{A} + j\mathcal{B}\right) \qquad (2.2.1)$$

where:

$$\mathcal{A} = \tfrac{1}{2} \log \{(\mathrm{Re}\ \chi)^2 + (\mathrm{Im}\ \chi)^2\} \tag{2.2.2}$$

$$\mathcal{B} = \tan^{-1} \left\{\frac{\mathrm{Im}\ \chi}{\mathrm{Re}\ \chi}\right\} \tag{2.2.3}$$

The quantity $\mathcal{A}$ in (2.2.1) represents the amplitude (modulation) distortion of the output signal which will not be considered further in this analysis. The quantity $\mathcal{B}$ represents the phase distortion. The total phase of the output signal is from (2.1.18), (2.2.1) and (2.2.3):

$$\theta = \omega_0 (\tau - a_1) + a_0 + m \sin p\, \omega_B \tau + \mathcal{B} \tag{2.2.4}$$

This involves rather tedious multiplication of series like those in (2.2.1) and leads to a result in the form

$$\begin{aligned} \mathcal{B} = {} & C_1 \cos p\, \omega_B \tau + S_1 \sin p\, \omega_B \tau \\ & + C_2 \cos 2p\, \omega_B \tau + S_2 \sin 2p\, \omega_B \tau \\ & + \dots \end{aligned} \tag{2.2.5}$$

This expression is directly applicable to phase modulation (see below); the corresponding expression for frequency modulation, in terms of instantaneous frequency, is found in the usual way by differentiating with respect to τ:

$$\begin{aligned} \Omega = \omega_0 + \Delta \{ & (1 + Hc_1) \cos p\, \omega_B \tau + Hs_1 \sin p\, \omega_B \tau \\ & + Hc_2 \cos 2p\, \omega_B \tau + Hs_2 \sin 2p\, \omega_B \tau \\ & + \dots \} \end{aligned} \tag{2.2.6}$$

where the harmonic ratios Hc_1, etc., can be expressed in terms of the signal parameters p and m and the coefficients of the expansions (2.1.6.7). These coefficients are tabulated below up to and including the term $s = 6$ in these expansions.

We note that the phase coefficients C, S of (2.2.5) can be obtained from the F.M. coefficients Hc, Hs using the following relations:

$$\begin{aligned} Hs_n &= -\frac{np\, \omega_B C_n}{mp\, \omega_B} = -\frac{n}{m} C_n \\ Hc_n &= \frac{np\, \omega_B S_n}{mp\, \omega_B} = \frac{n}{m} S_n \end{aligned} \tag{2.2.1}$$

$$\begin{aligned}
Hc_1 &= \tau_{20} p^2 + [\tau_{40} + \tau_{42} m^2]\, p_4 + [\tau_{60} + \tau_{62} m^2 + \tau_{64} m^4]\, p^6 \\
Hs_1 &= [\tau_{30} + \tau_{32} m^2]\, p^3 + [\tau_{50} + \tau_{52} m^2 + \tau_{54} m^4]\, p^5 \\
Hc_2 &= \sigma_{31} mp^3 + [\sigma_{51} + \sigma_{53} m^2]\, mp^5 \\
Hs_2 &= \sigma_{21} mp^2 + [\sigma_{41} + \sigma_{43} m^2]\, mp^4 + [\sigma_{61} + \sigma_{63} m^2 + {}_{65} m^4]\, mp^6 \\
Hc_3 &= \varphi_{42} m^2 p^4 + [\varphi_{62} + \varphi_{64} m^2]\, m^2 p^6 \\
Hs_3 &= \varphi_{32} m^2 p^3 + [\varphi_{52} + \varphi_{54} m^2]\, m^2 p^5 \\
Hc_4 &= \mu_{53} m^3 p^5 \\
Hs_4 &= \mu_{43} m^3 p^4 + [\mu_{63} + \mu_{65} m^2]\, m^3 p^6
\end{aligned}$$

where:

$$
\begin{aligned}
\tau_{20} &= b_2; \quad \tau_{30} = -(a_3 + a_2 b_1); \quad \tau_{32} = -\tfrac{3}{4} a_3 \\
\tau_{40} &= \tfrac{1}{2} (2 b_4 - a_2^2); \quad \tau_{42} = \tfrac{1}{4} (6 b_4 + b_1^2 b_2 - 2 a_2^2 - b_2^2 - 3 b_1 b_3) \\
\tau_{50} &= - (a_5 + a_4 b_1 + a_3 b_2 + a_2 b_3); \quad \tau_{44} = -\tfrac{5}{8} a_5 \\
\tau_{52} &= -\tfrac{1}{4} (15 a_5 + 6 a_4 b_1 + 17 a_3 b_2 + 17 a_2 b_3 - 8 a_3 b_1^2 - 9 a_2 b_1 b_2 + a_2 b_1^3) \\
\tau_{60} &= \tfrac{1}{2} (2 b_6 - 2 a_2 a_4 - a_3^2 - 2 a_2 a_3 b_1 - a_2^2 b_2) \\
\tau_{62} &= \tfrac{1}{4} (30 b_6 - 34 a_2 a_4 - 12 a_3^2 - 15 a_2 a_3 b_1 - 18 a_2^2 b_2 + 4 b_2 b_4 - 15 b_1 b_5 \\
&\qquad + 8 a_2^2 b_1^2 - b_2^3 + b_1^2 b_4 + 2 b_1 b_2 b_3 - 3 b_3^2) \\
\tau_{64} &= \tfrac{1}{16} (30 b_6 - 16 a_2 a_4 - 9 a_3^2 - 14 b_2 b_4 - 20 b_1 b_5 - 6 b_3^2 + 2 b_2^3 \\
&\qquad + 16 b_1 b_2 b_3 + 12 b_1^2 b_4 - 6 b_1^2 b_2^2 - 6 b_1^3 b_3 + 2 b_1^4 b_2) \\
\sigma_{21} &= -a_2; \quad \sigma_{31} = 3 b_3 - b_1 b_2; \quad \sigma_{41} = -(7 a_4 + 6 a_3 b_1 + 6 a_2 b_2 - a_2 b_1^2) \\
\sigma_{43} &= -a_4; \quad \sigma_{51} = 15 b_5 - 14 a_2 a_3 - 6 a_2^2 b_1 - b_1 b_4 - b_2 b_3 \\
\sigma_{53} &= \tfrac{1}{2} (10 b_5 - 6 a_2 a_3 - 4 b_2 b_3 - 6 b_1 b_4 + 2 b_1 b_2^2 + 3 b_1^2 b_3 - b_1^3 b_2) \\
\sigma_{61} &= -\tfrac{1}{3} (93 a_6 - 14 a_2^3 + 90 a_5 b_1 + 90 a_4 b_2 + 90 a_3 b_3 + 90 a_2 b_4 \\
&\qquad - 6 a_3 b_1 b_2 - 3 a_2 b_2^2 - 3 a_4 b_1^2 - 6 a_2 b_1 b_3) \\
\sigma_{63} &= -\tfrac{1}{3} (60 a_6 - 4 a_2^3 + 30 a_5 b_1 + 48 a_4 b_2 + 54 a_3 b_3 + 48 a_2 b_4 \\
&\qquad - 36 a_3 b_1 b_2 - 12 a_2 b_2^2 - 3 a_2 b_1^4 - 18 a_4 b_1^2 - 30 a_2 b_1 b_3 \\
&\qquad + 9 b_1^3 a_3 + 18 a_2 b_1^2 b_2) \\
\sigma_{65} &= -\tfrac{15}{16} a_6 \\
\varphi_{32} &= \tau_{32}; \quad \varphi_{42} = 3 \tau_{42}; \quad \varphi_{52} = -\tfrac{3}{4} (25 a_5 + 18 a_4 b_1 + 21 a_3 b_2 + 21 a_2 b_3 \\
&\qquad - 9 a_2 b_1 b_2 - 6 a_3 b_1^2 + a_2 b_1^3) \\
\varphi_{54} &= \tfrac{3}{2} \tau_{54}; \quad \varphi_{64} = \tfrac{9}{2} \tau_{64} \\
\varphi_{62} &= \tfrac{1}{4} (270 a_6 - 246 a_2 a_4 - 126 a_3^2 - 279 a_2 a_3 b_1 - 102 a_2^2 b_2 - 24 b_2 b_4 \\
&\qquad + 45 b_1 b_5 - 18 a_2^2 b_1^2 - 9 b_3^2 + 6 b_1 b_2 b_3 - 3 b_1^2 b_4 + b_2^3) \\
\mu_{43} &= \tfrac{1}{2} \sigma_{43}; \quad \mu_{53} = \sigma_{53}; \quad \mu_{65} = \tfrac{5}{2} \tau_{65} \\
\mu_{63} &= -\tfrac{1}{2} (45 a_6 - 2 a_2^3 + 20 a_5 b_1 + 32 a_4 b_2 - 30 a_3 b_1 b_2 + 32 a_2 b_4 + 34 a_3 b_3 \\
&\qquad - 18 a_4 b_1^2 - a_2 b_1^4 + 12 a_2 b_1^2 b_2 - 10 a_2 b_2^2 + 6 a_3 b_1^3 - 28 a_2 b_1 b_3)
\end{aligned}
$$

The practicability of this method of analysing the distortion depends of course on being able to express the amplitude and phase characteristics of the network in terms of the coefficients a, b of (2.1.6, 7). In Part III we shall develop a method for evaluating these coefficients for a large class of important networks.

PART III

THE EXPANSION OF THE IMPEDANCE FUNCTION

Most passive networks used in *FM* transmission are band-pass filters such that their impedances can be expressed in terms of the " band-pass " parameter.

$$X = Q \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) \tag{3.1.1}$$

For example, the impedance of a single tuned circuit can be written

$$Z(X) = (1 + X^2)^{-\frac{1}{2}} \exp (j \tan^{-1} X) \tag{3.1.2}$$

and expanded in the form (2.1.6, 7)

$$M(X) = 1 - \tfrac{1}{2} X^2 + \tfrac{3}{8} X^4 - \dots \tag{3.1.3}$$

$$\varphi(X) = X - \tfrac{1}{3} X^3 + \tfrac{1}{5} X^5 - \dots \tag{3.1.4}$$

However, to use the method described in (2.1) for the calculation of distortion, the expansion required must be in powers of $\omega - \omega_0 = np\,\omega_B$. To facilitate this we shall now develop a general method such that if the expansion in X can be obtained for any particular network, the corresponding expansion in $np\,\omega_B = \delta\omega$ can be determined immediately from tables.

The expansion will be made in such a way that the effects of mistuning can be readily determined. To this end we shall write

$$X = Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = Q\left(\frac{1+v}{\alpha} - \frac{\alpha}{1+v}\right) = Q\,\frac{K + 2v + v^2}{\alpha\,(1+v)} \tag{3.1.5}$$

when:

$$v = \frac{\delta\omega}{\omega_0} = \frac{\delta\omega}{2\,\omega_B}\cdot\frac{1}{Q}\,;\ \alpha = 1 + \delta;\ K = 1 - \alpha^2 \tag{3.1.6}$$

and α is the mistuning factor, i.e. the centre frequency is taken as $\alpha\,\omega_0$ rather than ω_0.

We can expand (3.1.5) in terms of the "small parameter" v:

$$X = \frac{Q}{\sqrt{1-K}}\left\{K + (2-K)\,v - (1-K)\sum_{s=0}^{\infty} (-)^s\, v^{s+2}\right\} \tag{3.1.7}$$

$$= \frac{Q}{\sqrt{1-K}}\sum_{s=0}^{\infty} \xi_{1s}\, v^s = \frac{1}{\sqrt{1-K}}\sum_{s=0}^{\infty} \frac{\xi_{1s}\,Q^{1-s}}{2^s}\left(\frac{\delta\omega}{\omega_B}\right)^s \tag{3.1.8}$$

Similarly, any power of X can be expressed as:

$$X^r = \frac{1}{(1-K)^{r/2}}\sum_{s=0}^{\infty}\frac{\xi_{rs}\,Q^{r-s}}{2^s}\left(\frac{\delta\omega}{\omega_B}\right)^s \tag{3.1.9}$$

Consider now either the phase or the amplitude of an impedance function; these can be represented as:

$$\begin{aligned}\varphi(X) &= A_0 + A_1 X + A_2 X^2 + \dots\\ M(X) &= 1 + M_1 X + M_2 X^2 + \dots\end{aligned} \tag{3.1.10}$$

The expansions required, i.e. in powers of $\left(\frac{\delta\omega}{\omega_B}\right)$ are in the form

$$\begin{aligned}\varphi\left(\frac{\delta\omega}{\omega_B}\right) &= a_0 + a_1\left(\frac{\delta\omega}{\omega_B}\right) + a_2\left(\frac{\delta\omega}{\omega_B}\right)^2 + \dots\\ M\left(\frac{\delta\omega}{\omega_B}\right) &= b_0 + b_1\left(\frac{\delta\omega}{\omega_B}\right) + b_2\left(\frac{\delta\omega}{\omega_B}\right)^2 + \dots\end{aligned} \tag{3.1.11}$$

Hence, comparing the coefficients in (3.1.11) and (3.1.10) and using the expansion (3.1.9), we find:

$$a_r = \frac{1}{2^r}\sum_{s=0}^{q}\frac{Q^{s-r}}{(1-K)^{s/2}}\,\xi_{sr}\,A_s;\quad b_r = \frac{1}{2_r}\sum_{s=0}^{q}\frac{Q^{s-r}}{(1-K)^{s/2}}\,\xi_{sr}\,M_s \tag{3.1.12}$$

TABLE I

$$\text{Let: } F(X) = F_0 + F_1 X + F_2 X^2 + \ldots\ldots F_6 X^6 + \ldots\ldots ,$$

where the expansion is stopped at X^6. The table below gives corresponding expansion in $(\delta\omega/\omega_B)$. The coefficients of given F_s are found from (3.1.8.9) while the polynomials of $K(=1-\alpha^2$, where α is the detuning parameter) are carried up to K, except where K enters the value as $(QK)^s = \eta^s$.

$$F\left(\frac{\delta\omega}{\omega_B}\right) = F_0 + \left\{F_1\eta\alpha^{-1} + F_2\eta^2\alpha^{-2} + F_3\eta^3\alpha^{-3} + F_1\eta^4\alpha^{-4} + F_5\eta^5\alpha^{-5} + F_6\eta^6\alpha^{-6}\right\}$$

$$+ \left(\frac{\delta\omega}{\omega_B}\right)\frac{1}{2}\left\{F_1(2-K)\alpha^{-1} + F_2\eta(4-2K)\alpha^{-2} + F_3\eta^2(6-3K)\alpha^{-3} + F_4\eta^3(8-4K)\alpha^{-4} + F_5\eta^4(10-5K)\alpha^{-5} + F_6\eta^5(12-6K)\alpha^{-6}\right\}$$

$$+ \left(\frac{\delta\omega}{\omega_B}\right)^2\frac{1}{4}\left\{F_1\frac{(-1+K)}{\alpha Q} + F_2(4-6K)\alpha^{-2} + F_3\eta(12-15K)\alpha^{-3} + F_4\eta^2(24-28K)\alpha^{-4} + F_5\eta^3(40-45K)\alpha^{-5} + F_6\eta^4(60-66K)\alpha^{-6}\right\}$$

$$+ \left(\frac{\delta\omega}{\omega_B}\right)^3\frac{1}{8}\left\{F_1\frac{(1-K)}{\alpha Q^2} + F_2\frac{(-4+8K)}{\alpha^2 Q} + F_3(8-24K)\alpha^{-3} + F_4\eta(32-72K)\alpha^{-4} + F_5\eta^2(80-160K)\alpha^{-5} + F_6\eta^3(160-300K)\alpha^{-6}\right\}$$

$$+ \left(\frac{\delta\omega}{\omega_B}\right)^4\frac{1}{16}\left\{F_1\frac{(-1+K)}{\alpha Q^3} + F_2\frac{(5-10K)}{\alpha^2 Q^2} + F_3\frac{(-12+39K)}{\alpha^3 Q} + F_4(16-80K)\alpha^{-4} + F_5\eta(80-280K)\alpha^{-5} + F_6\eta^2(240-720K)\alpha^{-6}\right\}$$

$$+ \left(\frac{\delta\omega}{\omega_B}\right)^5\frac{1}{32}\left\{F_1\frac{(1-K)}{\alpha Q^4} + F_2\frac{(-6+12K)}{\alpha^2 Q^3} + F_3\frac{(18-57K)}{\alpha^3 Q^2} + F_4\frac{(-32+152K)}{\alpha^4 Q} + F_5(32-240K)\alpha^{-5} + F_6\eta(192-960K)\alpha^{-6}\right\}$$

$$+ \left(\frac{\delta\omega}{\omega_B}\right)^6\frac{1}{64}\left\{F_1\frac{(-1+K)}{\alpha Q^5} + F_2\frac{(7-14K)}{\alpha^2 Q^4} + F_3\frac{(-25+78K)}{\alpha^3 Q^3} + F_4\frac{(48-252K)}{\alpha^4 Q^2} + F_5\frac{(-80+520K)}{\alpha^5 Q} + F_6(64-668K)\alpha^{-6}\right\}$$

$$\eta = QK = Q(1-\alpha^2) = -2Q\delta$$

so that the coefficients required when calculating distortion from (2.2.6) are obtainable using (3.1.12). To avoid the considerable labour involved, Table I has been prepared using (3.1.12) so that the a_s and b_s coefficients may be read directly.

In Table I the coefficients F_n are the A_n or M_n (i.e., the phase and amplitude coefficients, in the X-expansion) of (3.1.10). The successive rows represent a_s or b_s associated with a given power of $\left(\frac{\delta\omega}{\omega_B}\right)$. In the preparation of Table I we have neglected the nonlinear terms in K (see 3.1.6) since in most practical applications $|K| \leqslant 0.02$.

It should be observed that even if coefficients of powers of X in (3.1.10) higher than the sixth are included in the analysis, their inclusion would not modify any of the coefficients evaluated in Table I.

Examination of Table I shows that the most significant terms lie either on the main diagonal or immediately to the left and to the right of the diagonal terms. Those on the left (proportional to Q^{-1}) are more important in wide-band work, while those on the right (proportional to the detuning factor $\eta \doteq -2Q\delta$, where $\alpha = 1 + \delta$) are more important in narrow-band work where the effects of mistuning are particularly serious.

PART IV

APPLICATION

4.1. General Procedure of Using the Method

As an example of the technique described we shall evaluate the distortion introduced by a maximally-flat triple tuned circuit.

In the interest of simplicity we shall impose the following restrictions:

$$Q > 3;\ |\eta| = |2Q\delta| < 0.2;\ |\delta| < 0.01$$

$$p < 0.3;\ \Delta\omega < 0.3 \tag{4.1.1}$$

In spite of these restrictions it will be found that the resulting formulae have considerable practical value.

For the maximally flat triple tuned circuit.

$$M(X) = 1 - \tfrac{1}{2}X^6 \ldots$$

$$\varphi(X) = -2X - \tfrac{1}{3}X^3 - \tfrac{2}{5}X^5 \ldots \tag{4.1.2}$$

Using the values in Table I for the coefficients in (4.1.2) the a_s, b_s coefficients are found to have the approximate values shown in Table II. This table also shows the maximum numerical values of the coefficients a_s, b_s when the restrictions of (4.1.1.) apply.

Substituting from Table II into (2.2.6), the distortion formulae of Table III are obtained. In deriving these formulae only those terms have been retained which are necessary for the formulae to be accurate to within about $\pm$ 3 db.

4.3. Maximally Flat Amplitude Triple Tuned Circuit

The type of network considered here, namely the maximally flat amplitude triple tuned circuit, is of great practical importance in *FM* engineering because of comparatively easy tuning, good value of the gain-bandwidth product and relative simplicity of construction.

TABLE II

Approximate Forms of the coefficients a_s and b_s for the Maximally Flat Triple Tuned Circuit.

	Coeff.	Algebraic Form	Magn. of Coeff. Low Q	Magn. of Coeff. High Q
Phase Resp.	a_1	$-2 \quad -4\,(\delta Q)^2$	-2	$-2{\cdot}16$
	a_2	$1/2Q \quad +2\delta\,Q$	$0{\cdot}25$	$\pm 0{\cdot}4$
	a_3	$-1/3 \quad -16\,(\delta Q)^2$	$-0{\cdot}33$	$-1{\cdot}0$
	a_4	$1/4Q \quad +4\delta\,Q$	$0{\cdot}12$	$\pm 0{\cdot}8$
	a_5	$-2/5$	$-0{\cdot}40$	$-0{\cdot}40$
	a_6	$1/2Q$	$0{\cdot}25$	0
Ampl. Resp.	b_1	$-30\,(\delta Q)^2$	0	$-1{\cdot}2$
	b_5	$6\delta Q$	0	$\pm 1{\cdot}2$
	b_6	$-1/2$	$-0{\cdot}5$	$-0{\cdot}5$

N.B. $b_1 = b_2 = b_3 = 0$

From (4.1.2.) we see that the amplitude response of this network decreases as $1-\frac{1}{2}X^6$ near the mid-band frequency; we should thus expect the distortion of the fundamental component of the output signal to be negligible. The actual value of the fundamental distortion is so small for low values of Q, as to make impracticable any closer investigation. This can be seen immediately from the values of the coefficients Hc_1 and Hs_1 in Table III. For higher values of Q, the amplitude and phase of the fundamental component of the output follow the static expressions with small though significant deviations; this subject is discussed more fully in the next paragraph 4.3.1.

Considering the second harmonic, the second term for Hs_2 in Table III is $-2\delta Q\, m\, p^2$, as in the case of the single tuned network (see 4.2.3). Hence, to the first approximation, the amplitude of the second harmonic of the output signal is:

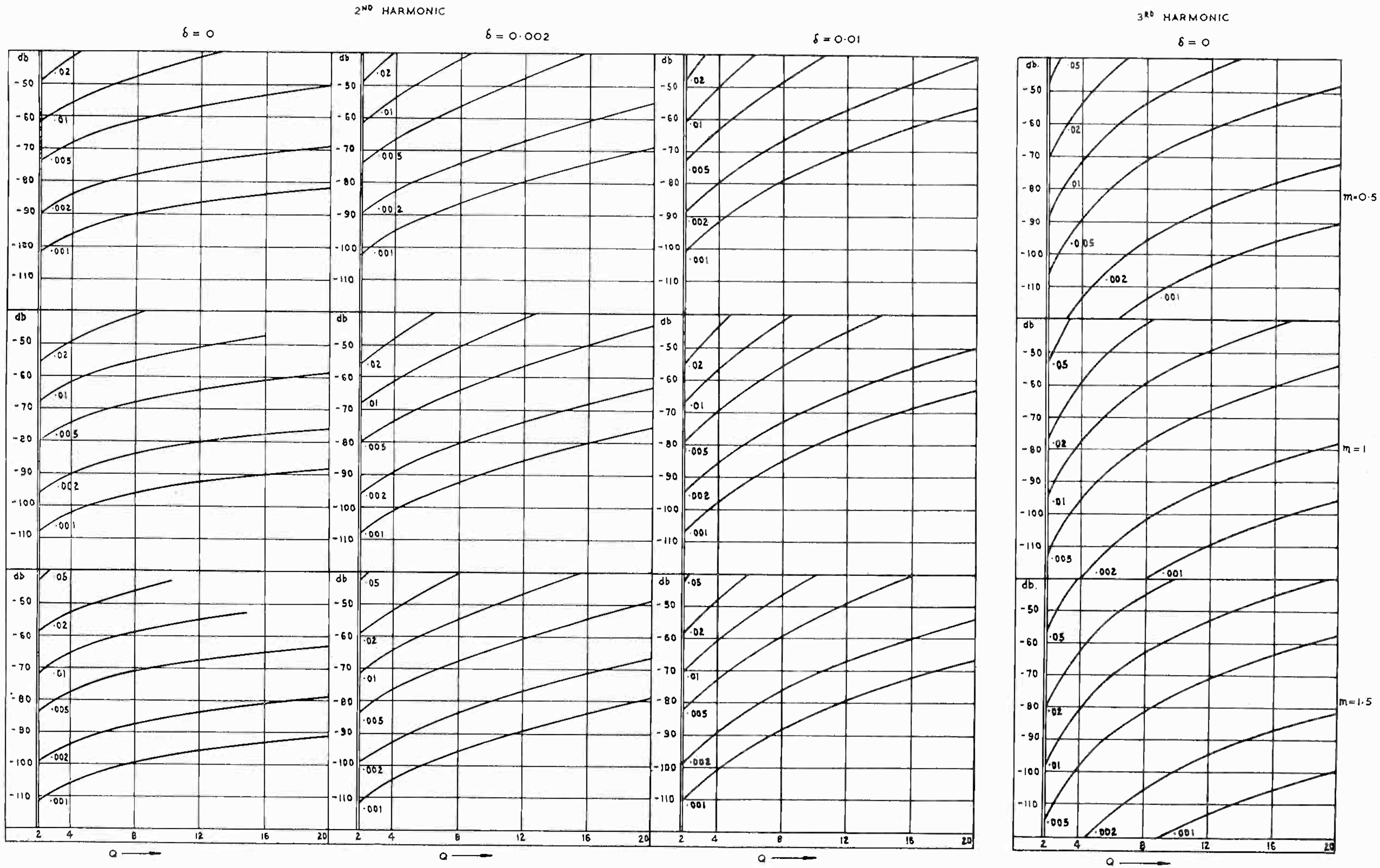
$$-8\,Q^3 \frac{\Delta\,\Omega.\ P.\ \delta\omega}{\omega_0^{\,3}} \qquad (4.3.1)$$

where $\Delta\,\Omega$ = maximum angular frequency deviation.

P = modulation angular frequency.

ω_0 = carrier angular frequency.

$\delta\,\omega$ = detuning angular frequency.



THE CURVES ARE IDENTIFIED BY THE APPROPRIATE VALUES OF $\frac{2\Delta \cdot \omega_0}{\omega_0}$ = $\frac{\text{MAX DEVIATION FREQUENCY}}{\text{CARRIER FREQUENCY}}$

FIG. 2

Maximally flat triple tuned circuit. FM response (without fundamental)

This expression is only valid for large values of Q. For $p > 0{\cdot}2$ the next term in Hs_2 becomes significant. Under these circumstances:

$$H_2 \doteqdot -8Q^3 \frac{\Delta\,\Omega.\,P.\,\delta\omega}{\omega_0^{\,3}} \left(1 + \frac{2\,P^2\,Q^2}{\omega_0^{\,2}}\right) \tag{4.3.2}$$

The third harmonic component can likewise be seen to be identical (as far as the first approximation is concerned) with that derived for the single tuned circuit (see 4.2.5 and Table III).

TABLE III.

Algebraic Forms of the coefficients for the Maximally Flat Triple Tuned Circuit.

$$2 \leqslant Q \leqslant 20$$
$$\delta Q \leqslant 0{\cdot}2$$

Fundamental	Hc_1	$-(\delta Q)^2\,(32 + 47\,m^2)\,p^4$
	Hs_1	$\frac{1}{12}\left[1 + 48\,(\delta Q)^2\right]\,(4 + 3\,m^2)\,p^3$
2nd Harm.	Hc_2	$\left\{\frac{1}{3}\left[\frac{7}{Q} + 298\,(\delta Q)\right] + \frac{1}{2}\left[\frac{1}{Q} + 64\,(\delta Q)\right]m^2\right\}mp^5$
	Hs_2	$-\frac{1}{2}\left[\frac{1}{Q} + 4\,(\delta Q)\right]mp^2 - \frac{1}{4}\left[\frac{1}{Q} + 16\,(\delta Q)\right](7+m^2)\,mp^4$
3rd Harm.	Hc_3	$-141\,(\delta Q)^2\,m^2\,p^4$
	Hs_3	$\frac{1}{4}\left[1 + 48\,(\delta Q)^2\right]m^2\,p^3$
4th Harm.	Hc_4	$\frac{1}{2}\left[\frac{1}{Q} + 64\,(\delta Q)\right]m^3\,p^5$
	Hs_4	$-\frac{1}{8}\left[\frac{1}{Q} + 16\,(\delta Q)\right]m^3\,p^4$

In Fig. 2 are plotted the amplitudes of the second and third harmonics against the Q of the circuit for various values of the maximum deviation frequency normalised with respect to the carrier-frequency. The values of $\Delta|\omega_0$ were again taken as: ·001, ·002, ·005, ·01, ·02. All these curves are plotted from expressions derived in Table III. These expressions are accurate as far as the p^4 term.

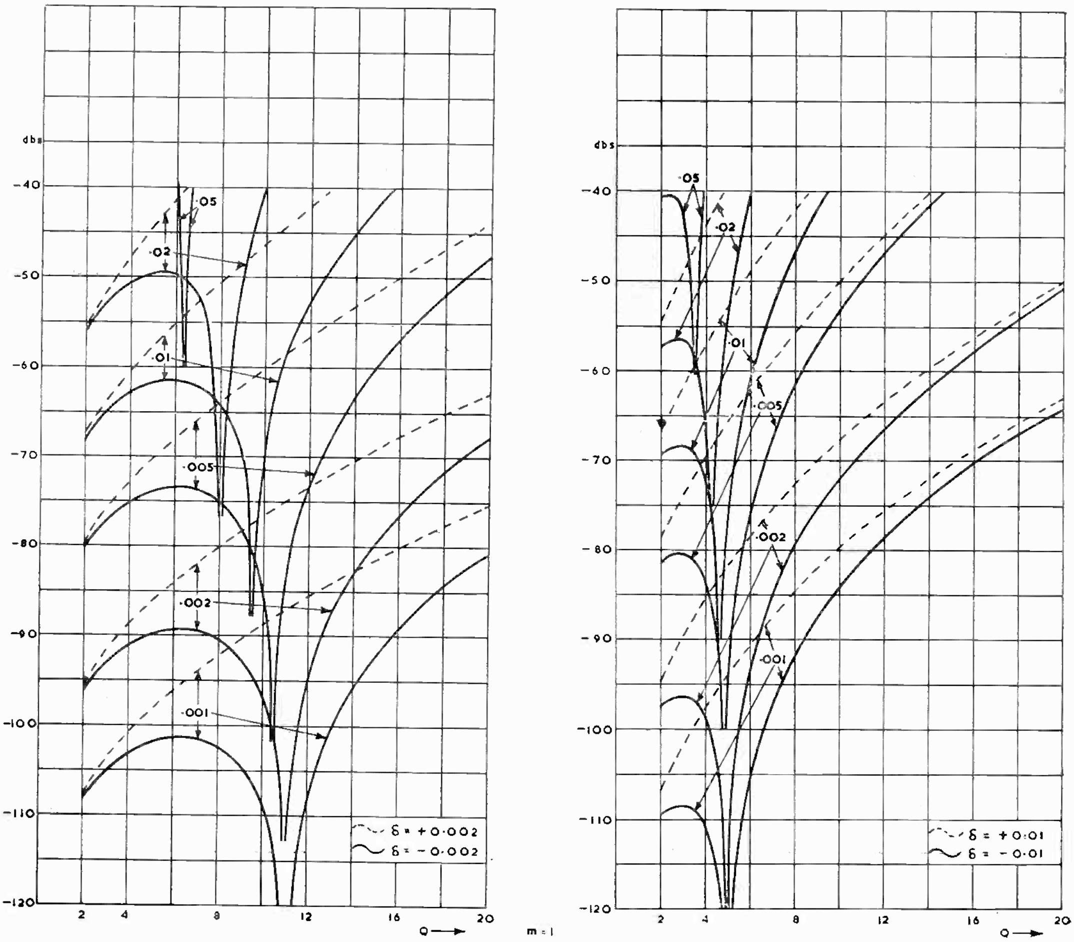
The curves in Fig. 2 are grouped in three rows corresponding to the values of the modulation index.

$$m = 0{\cdot}5,\ 1{\cdot}0,\ 1{\cdot}5$$

Furthermore, the second harmonic distortion is divided into three graphs for each value of m, corresponding to the positive detuning of

$$\delta = 0,\ 0{\cdot}002,\ 0{\cdot}01$$

It is seen that the positive detuning does not decrease the distortion here. In fact quite the opposite occurs, especially for large values of Q, where the second harmonic distortion for 1% detuning may be as much as 20 db greater than in the tuned case.



(*a*) (*b*)

FIG. 3

Maximally flat triple tuned circuit. Second harmonic distortion for negative detuning

(*a*) $\delta = \pm 0{\cdot}002$; (*b*) $\delta = \pm 0{\cdot}01$

This effect is not symmetrical, as for a negative detuning, a characteristic "dip" occurs in the distortion curves. In Figs. 3a and 3b we have shown the effects of negative detuning for the case of $m = 1$. In all cases the curve for positive detuning lies above the corresponding curve for negative detuning.

In Fig. 4 is shown the amount of negative detuning corresponding to the minimum second harmonic distortion as a function of Q of the network. This value of δ is not well defined as can be seen in Fig. 3a where the minima do not correspond to the same value of Q.

4.3.1. Distortion of the Fundamental Component over the Pass Band

Some work has been done previously by one of the authors (see preceding article) on the distortion of that component of the output voltage which is of the fundamental modulation frequency. The analysis was applicable to filters with symmetrical characteristic about the carrier, i.e. for sufficiently high Q so that the "band-pass parameter" X becomes proportional to the frequency deviation. Moreover the analysis could be managed only for a small number of sidebands; this necessarily limited the range of applicability of this method to values of the normalised modulation frequency p which were high enough to make $m = \dfrac{\Delta\omega}{p}$ sufficiently small.

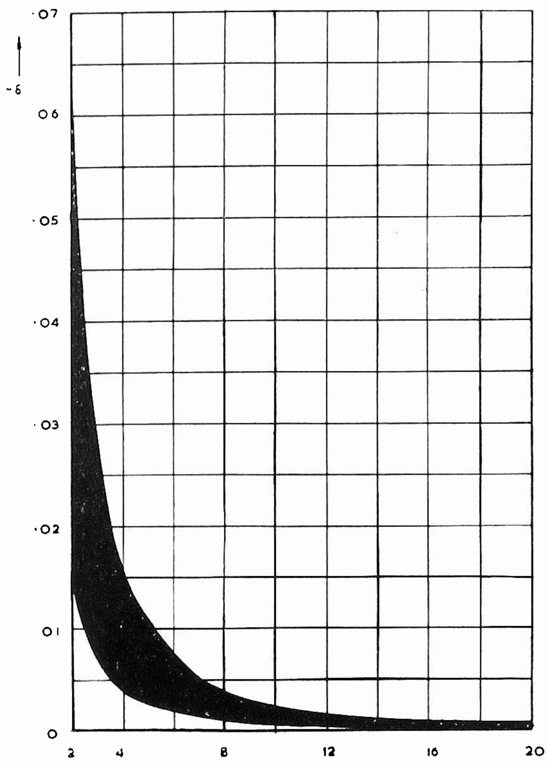


FIG. 4

Maximally flat triple tuned circuit. Amount of negative detuning for minimum second harmonic distortion

In particular the method was applied to a maximally flat amplitude triple-tuned circuit for which the frequency response is:

$$M(X) = (1 + X^6)^{-\frac{1}{2}} \qquad (4.3.1.1)$$

$$\varphi(X) = -\tan^{-1}\left\{\frac{2X - X^3}{1 - 2X^2}\right\} \qquad (4.3.1.2)$$

The range of the normalised modulation frequency taken was $0{\cdot}6 < p < 1{\cdot}0$ while the maximum deviation frequency Δ was varied from 0 to 1·0 (see pp. 106-110 in ref. 4). It was found then that the amplitude and phase distortions of the fundamental component of the output instantaneous frequency follow very closely the static response curves (see Fig. 9 and 10 in ref. 4).

The method of analysis discussed here, is, on the other hand, especially applicable for small values of p and thus it is of some interest to see how the results of the two methods fit each other over the pass band. In Fig. 5 we see the amplitude distortion curves together with the static selectivity curve (corresponding to $\Delta = 0$). The full lines on the right (large values of p) give the distortions obtained by the method used in ref. 4; those on the left (small values of p) give the distortions obtained by the method used here. Remembering the limits of the applicability of both

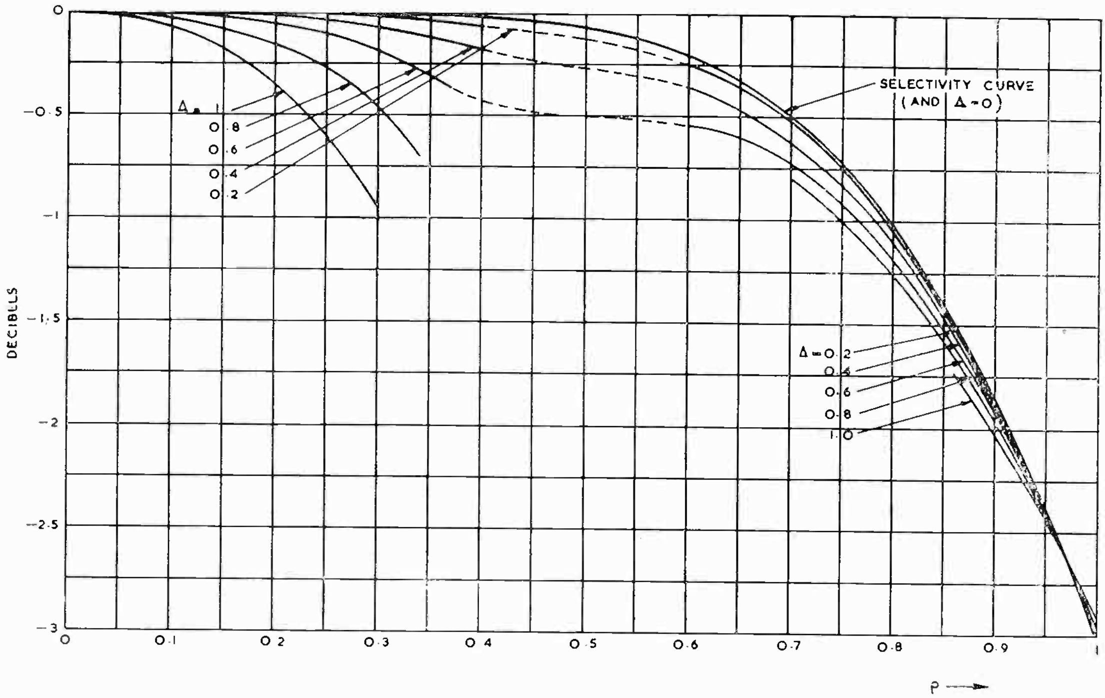


FIG. 5
Amplitude response of the fundamental for the maximally flat amplitude triple tuned circuit

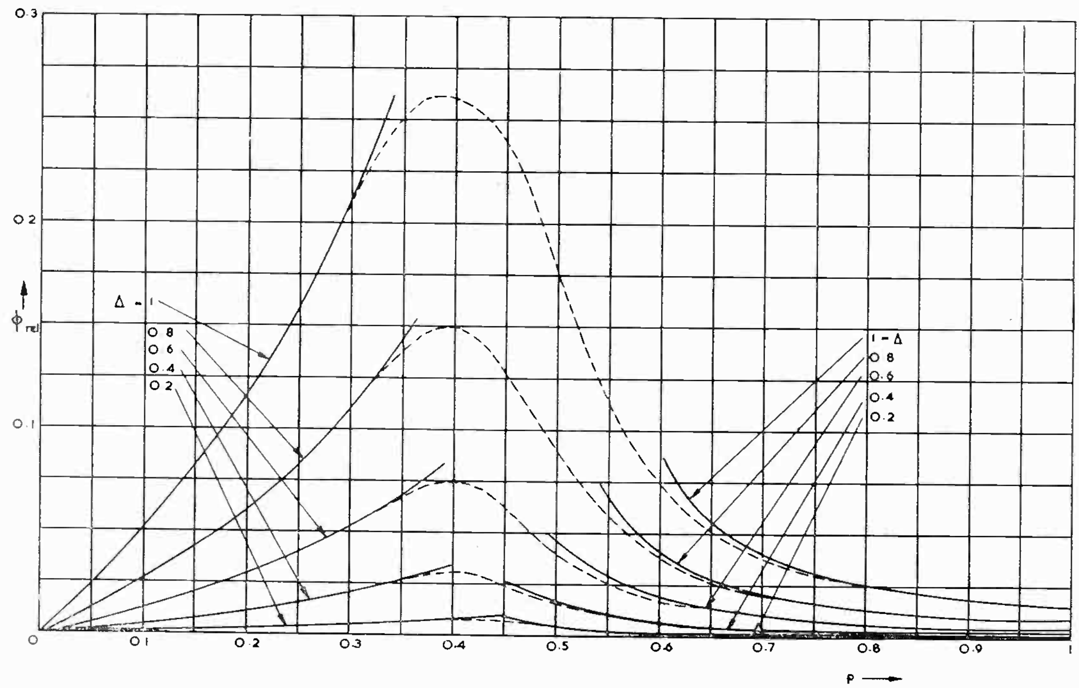


FIG. 6
Relative phase response of the fundamental for the maximally flat amplitude triple tuned circuit

methods (see pp. 119 and 121) it is easy to join the two sets of curves for the smaller values of the maximum deviation frequency Δ (i.e. $0 \leqslant \Delta \leqslant 0{\cdot}6$). The continuous curves, including the dashed portions of the middle give the fundamental distortion over the whole of the band. For higher values of Δ ($> 0{\cdot}6$) the joining of two sets of curves becomes more problematical because of the limited range of each of the two methods.

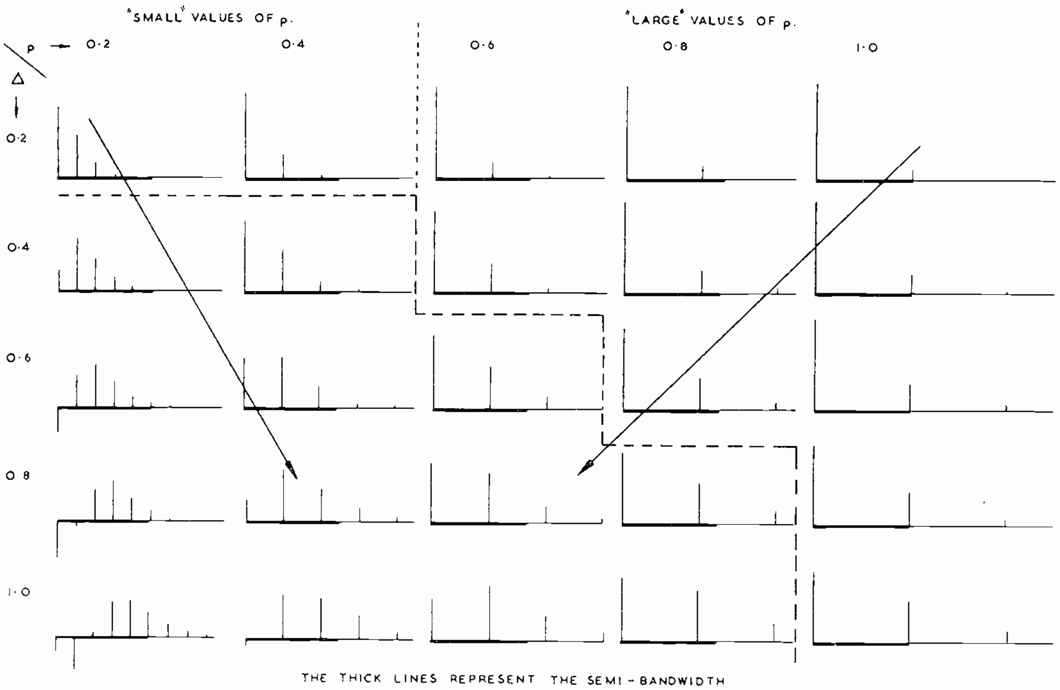


FIG. 7

The sideband distribution

Fig. 6 shows the two similar sets of curves joined together and representing the relative phase of the fundamental, where:

$$\varphi_{rel} = -\varphi_{static} + \varphi_{total} + 2\pi \tag{4.3.1.3}$$

where φ_{static} = the static phase characteristic of the network.

φ_{total} = the output phase characteristic of the fundamental.

Returning to Fig. 5 we observe a definite maximum of the amplitude distortion for values of p between 0·4 and 0·6 of the semi-bandwidth. This phenomenon can be explained by considering the sideband distribution associated with given values of p and Δ. In Fig. 7 we see the relative amplitudes and positions of the upper sidebands for given values of p and Δ, the semi-bandwidth being represented by the thick lines. The values of p and Δ which can be treated by the analysis used in ref. 4 are associated with the sideband distributions above the broken line; the dotted vertical line divides " large " values of p (treated in ref. 4) from " small " values of p. It will be observed that as we move in the direction of the arrows more and more of the significant sidebands are shifted to the regions of the passband where the approximation due to taking $q = 6$ in 2.1.12 results in significant errors (i.e. p and $\Delta > 0{\cdot}6$).

Fig. 6 also shows that the relative phase characteristic behaves in a similar way.

PART V

COMPARISON OF THE RESULTS OBTAINED HERE WITH OTHER METHODS

5.1. Standards of Comparison

In Part I of this article we have discussed the powerful method introduced by Fry and Carson and subsequently developed by Stumpers. We have seen there that the " quasi-stationary " approximation taken alone will give misleading results especially when applied to that part of the recovered signal which is of the fundamental modulation frequency (see section 1.1.2).

It will be useful now to compare the results obtained in Part IV with the results which would be obtained in similar conditions by Stumpers and by Fry and Carson using their " retarded time " formula.

We shall thus consider a network whose phase and amplitudes are expanded in terms of $\left(\frac{\delta\omega}{\omega_B}\right)$ as explained in Part III. The expansion will be carried out only to the cubic term inclusively.

Thus:

$$\begin{aligned} \varphi(\delta\omega) &= a_1\,\delta\omega + a_2\,(\delta\omega)^2 + a_3\,(\delta\omega)^3 \\ M(\delta\omega) &= 1 + b_1\,(\delta\omega) + b_2\,(\delta\omega)^2 + b_3\,(\delta\omega)^3 \end{aligned} \qquad (5.1.1)$$

It is important here to state briefly the criterion by which a method of evaluating the response of a network to a harmonic *FM* signal can be tested. In section 1.1.2 we have seen that the " quasi-stationary " approximation fails as far as the fundamental component of the output signal is concerned. The obvious test will thus be to see whether this fundamental follows the static response of the network for vanishingly small values of the modulation frequency p.

We have seen this to be the case in the previous part of this article. Moreover, in the case of the maximally flat triple-tuned circuit the fundamental output never deviates considerably from the static response within the whole of the pass-band (see section 4.3 and ref. 4).

The examination of (5.1.1) will show that for networks whose amplitude response is an even function of the frequency deviation $\delta\omega$, the most important coefficients are a_1 in the phase expansion and b_2 in the amplitude expansion.

The coefficient a_1 will cause the whole of the signal to be delayed by the time equal to a_1, since it represents the initial delay; the coefficient b_2 will thus be decisive as far as the distortion is concerned and in particular the fundamental part of the output signal should, for vanishingly small values of p, follow the expression:

$$\sqrt{1 + 2b_2\,p^2}$$

Remembering that b_2 is always negative for band-pass filters, it is clear that the output at fundamental frequency will diminish as p increases.

5.2. The Stumpers' Method.

The formula of Stumpers is given in ref. 2, equation 16a. It is easily seen that if the input frequency is given by:

$$h = \omega_0 + \Delta \sin p\,\omega_B\,t \tag{5.2.1}$$

the two terms in curled brackets of (16a) are respectively of orders p^3 and p^4. We shall only consider here terms up to p^3 and thus we can write Stumpers' formula (16a) as:

$$\omega_0 = h\,(p\,\omega_B\,t) + p\varphi\,'h' + p^2 \left[\tfrac{1}{2}\,h''\left(\varphi'^2 - \frac{M''}{M}\right)\right]\,.. \tag{5.2.2}$$

Here, from (5.1.1):

$$\begin{aligned} \varphi' &= a_1 + 2a_2\,\delta\omega + 3a_3\,(\delta\omega)^2 \\ M'' &= 2b_2 + 6b_3\,\delta\omega \end{aligned} \tag{5.2.3}$$

Note that the dashes here refer to differentiation.

After some manipulation, we obtain formulae corresponding to those given in Table II:

$$\begin{aligned} Hs_1 &= 1 + \tau'_{20}\,p^2 \\ Hc_1 &= \tau'_{10}\,p + \tau'_{32}\,m^2\,p^3 \\ Hs_2 &= \sigma'_{21}\,mp^2 \\ Hc_2 &= \sigma'_{31}\,mp^3 \\ Hc_3 &= -\,\tau'_{32}\,m^2\,p^3 \end{aligned} \tag{5.2.4}$$

where:

$$\begin{aligned} \tau'_{20} &= \tfrac{1}{2}\,(\underline{2b_2} - a_1^2) \\ \tau'_{10} &= a_1 \qquad \sigma'_{21} - \underline{a_2} \\ \tau'_{32} &= \underline{\tfrac{3}{4}\,a_3} \qquad \sigma'_{31} = 2a_1\,a_2 + \underline{b_1\,b_2} - \underline{3b_3} \end{aligned} \tag{5.2.5}$$

If we compare (5.2.5) with the coefficients obtained by our method we observe that they are identical as long as $a_1 = 0$; since the underlined coefficients are those obtained by us by the method described in Part II.

We see thus that as far as the 2nd and 3rd harmonics are concerned, Stumpers agrees with us provided that the first order approximation is taken; we have seen in Part IV that this approximation is quite sufficient for many practical applications.

Equally, for the fundamental, we can write approximately:

$$\begin{aligned} Hs_1 &\doteqdot 1 + \tau'_{20}\,p^2 \\ Hc_1 &\doteqdot \tau'_{10}\,p \end{aligned} \tag{5.2.6}$$

$$\begin{aligned} \therefore \quad \sqrt{(Hs_1)^2 + (Hc_1)^2} &\doteqdot \sqrt{1 + (2\tau'_{20} + \tau'^2_{10})\,p^2} \\ &= \sqrt{1 + 2b_2\,p^2} \end{aligned} \tag{5.2.7}$$

as required.

Hence, the Stumpers formula agrees completely with that obtained by us as far as the first approximation is concerned. Moreover, expression (5.2.7) shows exactly where the "quasi-stationary" approximation fails in neglecting further terms. By assuming that $\tau'_{20} = 0$. i.e. by neglecting the p^2 term one obtains a rising amplitude response of the fundamental, whereas in fact this coefficient cancels out with the corresponding one in the p^2 term leaving only the part depending on the static amplitude response, as required.

We have thus shown that it is a dangerous procedure to use the "quasi-stationary" approximation, since (as this has been shown in pp. 114 and 115, and in section 1.1.2) although this approximation will in general give the fundamental, component as a function of p, such dependence is completely misleading. This is due to the way in which the "quasi-stationary" approximation picks out the terms in the power expansion of the output signal (see p. 115).

On the other hand if one has to use the "quasi-stationary" approximation, the only way one can arrive at a reasonable result is to change the time scale as in (2.1.8). This leads us to the "retarded time" formula of Fry and Carson, but this is no longer "quasi-stationary" in form.

It is interesting to note that in a recent article on the same subject, R. G. Medhurst[5] comes to similar conclusions (see pp. 173, 175 of ref. 5) stating that "the constant part of the group delay (i.e. a) should not affect the distortion". This is certainly the case, although not for the reasons stated by the author, as can be seen from the above discussion.

The Fry and Carson "retarded time" formula (35) is however much more involved to use especially if one tries to obtain the output instantaneous frequency; this requires evaluating the inverse tangent of the complex expression which is rather laborious. Stumpers expression (16a) gives the instantaneous frequency directly and is thus of much greater practical importance.

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